Updated 08/16/04

Advanced Functions and Modeling

Advanced Functions and Modeling provides students an in-depth study of modeling and applying functions. Home, work, recreation, consumer issues, public policy, and scientific investigations are just a few of the areas from which applications should originate. Appropriate technology, from manipulatives to calculators and application software, should be used regularly for instruction and assessment.

Prerequisites

- ***Describe graphically, algebraically and verbally phenomena as functions; identify independent and dependent quantities, domain, and range, and input/output.***
- ***Translate among graphic, algebraic, numeric, and verbal representations of relations.***
- ***Define and use linear, quadratic, cubic, and exponential to model and solve problems.***
- ***Use systems of two or more equations or inequalities to solve problems.***
- ***Use the trigonometric ratios to model and solve problems.***
- ***Use logic and deductive reasoning to draw conclusions and solve problems.***

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GOAL 1: The learner will analyze data and apply probability concepts to solve problems.

1.01 Create and use calculator-generated models of linear, polynomial, exponential, trigonometric, power, and logarithmic functions of bivariate data to solve problems.
a) Interpret the constants, coefficients, and bases in the context of the data.
b) Check models for goodness-of-fit; use the most appropriate model to draw conclusions and make predictions.

1.02 Summarize and analyze univariate data to solve problems.
a) Apply and compare methods of data collection.
b) Apply statistical principles and methods in sample surveys.
c) Determine measures of central tendency and spread.
d) Recognize, define, and use the normal distribution curve.
e) Interpret graphical displays of univariate data.
f) Compare distributions of univariate data.

1.03 Use theoretical and experimental probability to model and solve problems.
a) Use addition and multiplication principles.
b) Calculate and apply permutations and combinations.
c) Create and use simulations for probability models.
d) Find expected values and determine fairness.
e) Identify and use discrete random variables to solve problems.
f) Apply the Binomial Theorem.

GOAL 2: The learner will use functions to solve problems.

2.01 Use logarithmic (common, natural) functions to model and solve problems; justify results.
a) Solve using tables, graphs, and algebraic properties.
b) Interpret the constants, coefficients, and bases in the context of the problem.

2.02 Use piecewise-defined functions to model and solve problems; justify results.
a) Solve using tables, graphs, and algebraic properties.
b) Interpret the constants, coefficients, and bases in the context of the problem.

2.03 Use power functions to model and solve problems; justify results.
a) Solve using tables, graphs, and algebraic properties.
b) Interpret the constants, coefficients, and bases in the context of the problem.

2.04 Use trigonometric (sine, cosine) functions to model and solve problems; justify results.
a) Solve using tables, graphs, and algebraic properties.
b) Create and identify transformations with respect to period, amplitude, and vertical and horizontal shifts.
c) Develop and use the law of sines and the law of cosines.

2.05 Use recursively-defined functions to model and solve problems.
a) Find the sum of a finite sequence.
b) Find the sum of an infinite sequence.
c) Determine if a given series converges or diverges.
d) Translate between recursive and explicit representations.

Vocabulary Concepts Skills

Regression

Residuals

Correlation Coefficient (linear data)

R^2

Calculator Limitations (with respect to data)

Translation of Data

Interpret Constants, Coefficients, Bases

Select Best Model

Interpolate

Extrapolate

Estimate

Predict

DRAFT

1.01 Create and use calculator-generated models of linear, polynomial, exponential, trigonometric, power, and logarithmic functions of bivariate data to solve problems.

A. A ball is dropped over a motion detector and its height is recorded. The height is measured in feet and the time in seconds. The data is shown in the table below. Find a quadratic model for this data. Use the method of completing the square to write your model in the form $y = k(x - h)^2 + v$. Are **k** and **v** reasonable in the context of the scenario? Explain.

Time	0	0.04	0.08	0.12	0.16	0.20	0.24	0.28	0.32	0.36	0.40
Height	4.54	4.46	4.34	4.16	3.94	3.68	3.37	3.02	2.63	2.2	1.74

B. In December 2003, a significant ice storm struck North Carolina resulting in numerous power outages. Because of the cold, wet weather and the extraordinarily large number of outages, it took many days for Duke Power to restore electricity to its customers in Durham. The table below shows the number of days since the storm struck, the percent of customers whose electricity was restored that day, and the cumulative percent of customers whose power was restored.

Days	1	2	3	4	5	6
Restored Daily	0.16	9.71	19.91	21.62	16.8	13.64
Total Restored	0.16	9.87	29.78	51.4	68.2	81.84
Days	7	8	9	10	11	
Restored Daily	9.58	6.1	1.74	0.73	0.01	
Total Restored	91.42	97.52	99.26	99.99	100	

Plot the data in the first and third rows. Which variable is independent and why? The data appear to level off. Why is that reasonable? Find an appropriate function to model the data. What is your equation? What do the coefficients in the model mean in the context of the problem? Are the reasonable values?

Vocabulary
Concepts
Skills

C. Students were given a collection of 300 dice. The instructions were to roll all of the dice, let them land on the floor, and then remove the dice showing FIVE. The students were told to repeat this process, each time removing all the Five's, until there were fewer than 50 dice left. The results are shown below.

Roll	1	2	3	4	5	6	7	8	9	10
Dice Remaining	252	207	170	146	123	100	85	67	56	48

Find a model for the data. Based on the model, how many times would the students have to roll the dice so that fewer than ten dice remained?

D. If you have a set of balls, all of the same diameter, can you predict the number of balls that will fit in a trough? Graph the data shown.

Diameter (in)	$1\frac{3}{4}$	$2\frac{3}{8}$	$\frac{3}{4}$	$\frac{9}{16}$	$1\frac{5}{16}$	$1\frac{3}{8}$	$\frac{15}{16}$
Number	6	5	14	19	8	8	11

Which is the independent data? Why? What happens as the number of balls gets smaller? larger? How would you determine the size of the trough? Where is that on the graph? Which toolkit funtion does the data resemble? Determine a function of best fit and explain its selection. Define the coefficient(s) in terms of the data.

1.02 Summarize and analyze univariate data to solve problems.

Vocabulary
Concepts
Skills

Mean

Median

Variance

Standard Deviation

Normal Distribution

Random Sampling

Census

Survey

Bias

Population

Various Graphical Representations

Vocabulary Concepts Skills

Counting

Random

Event

Success

Trial

Sample Space

Dependent

Independent

Compound

Mutually Exclusive

Conditional

Binomial Probability

Expected Value

Random Variable

Fairness

Game Theory

Markov Chains

Simulation

Combination

Permutation

1.03 Use theoretical and experimental probability to model and solve problems.

A. The student population at Roosevelt High is 1046. The entire student population was surveyed, and then categorized according to class and number of hours worked per week at a paying job.

	0 hr.	Work < 10 hr.	Work10 to 20 hr.	Work > 20 hr.
Freshmen	240	13	2	1
Sophomores	223	52	4	0
Juniors	103	25	88	47
Seniors	58	35	110	45

What is the probability that a randomly selected student from this school is a senior who does not have a job?
What is the probability that a randomly selected student is a sophomore who works between 10 and 20 hours per week?
What is the probability that a randomly selected student is a freshman?
What is the probability that a randomly selected student does not have a job?
What is the probability that a student is a freshman OR works less than 10 hours per week?
Which events are mutually exclusive?

- Being a freshman and working less then 10 hours per week.
- Being a senior and not having a job.
- Being a sophomore and working more than 20 hours per week.

What is the probability that a randomly selected student works more than 20 hours per week?
What is the probability that a randomly chosen student works more than 20 hours per week, given that s/he is a freshman?
What is the probability that a randomly chosen student works more than 20 hours per week, given that s/he is a senior?
Based on your last two answers, what comparison can you make between the freshman class and the senior class?
Determine whether or not a student's class (freshman, sophomore, junior or senior) and a students work hours are independent of each other. Base your answer on the definition of independent events.

Vocabulary
Concepts
Skills

B. While playing Monopoly, a player can gain money, lose money, or have no change in money when he takes his turn. Suppose that a particular stage of a game is such that a player will lose $1000 if he rolls a two, three or five because those numbers will cause him to land on an opponent's property. If he rolls a seven or higher he will gain $200 because he will pass GO. If he rolls any other numbers, there will be no change in the amount of money he has. Find the expected value of the change in the amount the player has.

C. A fair coin is tossed five times. On each toss, the probability of a head is
$^1/_2$, and the five tosses are all independent events. What is the probability that two of the five coin tosses produce a head? What is the probability that the five coin tosses produce at least one head? At most one head? What is the expected value of the number of heads?

D. An unfair coin is weighted so that the probability of a head is $^1/_3$ and the probability of a tail is $^2/_3$. The coin is tossed seven times, and the outcome on each toss is independent of that on all of the other tosses. What is the probability that the seven coin tosses produce at least two heads? Exactly two heads? Which is more likely, two heads of out seven or four heads out of seven? Justify your answer.

E. Suppose two random numbers are chosen between zero and eight. What is the probability that their sum is less than or equal to five?

F. Each day two out of three teams are randomly selected to participate in a game. What is the probability that team A is selected on at least two of the next three days?

G. The local observatory said that a meteor shower could be observed tonight. According to one astronomer, there is a 90% chance of seeing a shooting star between 10:00 and 11:00 PM. Given that the probability of seeing a shooting star is constant throughout the hour, what is the probability of seeing shooting star between 10:00 and 10:10 PM?

Vocabulary
Concepts
Skills

Graph

Independent

Dependent

Domain

Range

Coefficients

$y = a \cdot \log(bx + c) + d$

$y = a \cdot \ln(bx + c) + d$

Zeros

Intercepts

Asymptotes

Minimum

Maximum

Increasing

Decreasing

Laws of Exponents

Laws of Logarithms

Global versus Local Behavior

Continuous

Discrete

Solve Equations, Justifying Steps Used

2.01 Use logarithmic (common, natural) functions to model and solve problems; justify results.

A. A study showed that the function $M(t) = 6\log(2t + 4)$ approximates the population of mice in an abandoned building where t is the number of months since the building was abandoned five years before. Identify the 12-month interval when the mice population grew the most; the least.

B. An investment earning 8.5 % annually can be evaluated after t years, using $A(t) = A_0 e^{(.085t)}$ When will a $1200 investment be valued at $1600? At what rate would it double in 6 years?

C. In January 2003, Mexico experienced a major earthquake which registered 7.6 on the Richter scale. The Richter scale was revised in 1979 so that R, the magnitude of the earthquake, is defined by $R = 0.67\log(0.37E) + 1.46$ where E is energy in kilowatt-hours. How much energy was released in this earthquake?

D. On the decibel scale, the faintest sound a human can hear is defined as zero decibels. A sound 10 times more powerful than "faintest" is assigned a noise level of 10 decibels, a sound 100 times more powerful is assigned 20 decibels, a sound 1000 times more powerful is assigned 30 decibels, and so on. If N represents the noise level in decibels and P represents the number of times more powerful than "faintest", write the function that gives N in terms of P. During the 2002 Stanley Cup Finals in Raleigh, the noise level reached 116 decibels in the arena. How many more times powerful was the arena noise than the "faintest" sound?

Vocabulary
Concepts
Skills

Graph

Independent

Dependent

Domain

Range

Minimum

Maximum

Increasing

Decreasing

Global vs. Local Behavior

Continuous

Discrete

System of Equations

Solve Equations, Justifying Steps Used

2.02 Use piecewise-defined functions to model and solve problems; justify results.

A. Sketch the graph and state the domain, range, and zeros of $h(x)$.

$$h(x) = \begin{cases} x^2 & \text{if } -2 \le x < 1 \\ 2 - x & \text{if } 1 \le x < 4 \end{cases}$$

B. Construct tax functions for a single taxpayer (standard deduction, one exemption) using the most recent North Carolina and federal income tax schedules and Social Security (see **B-74** in ***Resources for Algebra***). Graph the functions. Determine the tax liability for gross incomes of $20,000, $30,000, and $100,000. A person's **effective tax rate** is defined as the percent of gross income that is paid in tax. What is the effective tax rate for each of the incomes mentioned? Construct effective tax rate functions based on the income tax schedules and Social Security. Create a composite tax function for a single taxpayer based on the income tax schedules and Social Security. Justify your results.

C. A fast-moving cold front in the Northeast can cause temperatures to drop very quickly then rise again. The following data uses t as the hours since midnight on a day the cold front moves in, and T as the temperature in degrees Fahrenheit.

t	0	1	2	3	4	5	6	7	8	9	10
T	3	1	–1	–3	–5	–7	–5	–3	–1	1	3

Make a sketch of this data. Could this data have a linear model? Explain. Could it be quadratic? Explain. Could you use a piecewise function to model it? What would it be? The absolute value function has a similar shape. Use a transformation of $f(x) = |x|$ to model this data.

D. Each orange tree in a California grove produces 600 oranges per year if no more than 20 trees are planted per acre. For each additional tree planted per acre, the yield per tree decreases by 15 oranges. Describe the orange tree yield algebraically and determine how many trees per acre should be planted to obtain the greatest number of oranges.

2.03 Use power functions to model and solve problems; justify results.

Vocabulary
Concepts
Skills

Graph

Independent

Dependent

Domain

Range

Coefficients

$y = ax^b + c$

Zeros

Intercepts

Asymptotes

Minimum

Maximum

Increasing

Decreasing

Laws of Exponents

Laws of Logarithms

Global versus Local Behavior

Continuous

Discrete

Solve Equations, Justifying Steps Used

Vocabulary
Concepts
Skills

Graph

Independent

Dependent

Domain/Range

Period

Amplitude

Phase Shift

Frequency

Coefficients

$y = a \cdot \sin(bx + c) + d$

$y = a \cdot \cos(bx + c) + d$

Intercepts

Law of Sines

Law of Cosines

Unit Circle

Radian/Degree Measure

Special Angles (multiples of π, $\pi/2$, $\pi/3$, $\pi/4$, $\pi/6$)

Solve Equations, Justifying Steps Used

2.04 Use trigonometric (sine, cosine) functions to model and solve problems; justify results.

A. Sketch the graph of $f(x) = a \cdot \cos(bx) + c$ for $0 \le x \le 2\pi$ where a = 5, b = 2, and c = 3. Identify the intercepts and the maximum and minimum points. If **a** changes value and gets close to zero, how does the graph of $f(x)$ change? If **b** increases, how does the graph of $f(x)$ change?

B. At a particular location on the Atlantic coast, a pier extends over the water. The height of the water on one of the supports is 5.4 feet, at low tide (2 AM) and 11.8 feet at high tide, 6.2 hours later. Write an equation describing the depth of the water at this location t hours after midnight.

Use the form $h(t) = a\cos\left[\frac{2\pi}{T}(t-b)\right] + c$

What will be the depth of the water at this support at 4 AM?

C. Maximum and minimum average daily temperatures of two cities are given. On the same graph, sketch a sinusoidal curve (day of the year, temperature) for each city and create an equation to represent each curve. Explain differences between the curves.

	January 15 (15th day)	July 16 (197th day)
Montreal, Quebec	-10°C	21°C
Orlando, Florida	15°C	28°C

D. The height of a seat on a Ferris wheel with a diameter of 14 meters, t seconds after it begins to turn at 3 rpm, can be computed using this sinusoidal model, $H(t) = 7\cos\left[\frac{2\pi}{20}(t-10)\right] + 8$. Graph the function.

When will the height of the seat be a minimum? A maximum? What are these heights? If the wheel is replaced with one with a larger diameter, 16 feet,
how would the parameters in the equation be affected? Test your conjecture on the calculator. If the original wheel turns at 4 rpm, which parameter(s) are affected? Graph your altered equation to check the outcome.

Vocabulary
Concepts
Skills

Linear Sequence

Geometric Sequence

Geometric Series

Subscript Notation

Summation Notation

Converge

Diverge

Translate between Recursive and Explicit Representations

2.05 Use recursively-defined functions to model and solve problems.

A. The Tower of Hanoi puzzle consists of a stack of wooden disks of graduated sizes on one of three wooden pins. One may move only one disk at a time, and never put a larger disk onto a smaller disk. The goal is to move all the disks to another pin in a minimum number of moves. The number of moves needed to relocate n disks is represented by $S(n)$.

n	1	2	3	4	5	6	7	8
$S(n)$	1	3	7	15	31			

Find a recursive function to describe this data of the form $S(n+1) = a\,S(n) + b$. Use this equation to complete the table.
There is an explicit formula to model this data as well.
It is of the form $S(n) = b^n + c$. Find this formula and confirm the results in your table. How many steps would it take to move 20 disks?

B. Mr. Smith has recently retired, and he will rely on his savings to supplement Social Security. He has an account in which the balance was $25,000 on June 1, when he retired. The account earns interest at the rate of 4.8% annual interest, and the interest is compounded monthly. Mr. Smith will withdraw $1100 from the account each month. Write a recursive system to represent the account balance over time. How much money will be in the account 12 months after Mr. Smith retired? How long will it take for the balance to drop to less than $5,000?

C. After a person takes pain medication, his kidneys filter the medicine out of his blood stream. During any four hour time period, his kidneys will remove 35% of the medication that remains in the bloodstream. Suppose a patient takes 800 mg of ibuprofen at 8 AM on Tuesday morning. If he does not repeat the dosage, how much medicine will remain his blood stream at 12 midnight on the same day?
Suppose a patient takes 800 mg of ibuprofen at 8 AM on Tuesday morning, and repeats the same dosage every 4 hours. How much medicine will remain his blood stream at 12 midnight on the same day (immediately before he takes his 5th dosage)?
This patient continues taking 800 mg of ibuprofen every 4 hours for several days. The amount of ibuprofen in his bloodstream (in mg) varies between two amounts. What are they?

D. For the sequence of numbers defined as $a_1 = 7$, $a_2 = \sqrt[3]{a_1} + 1$, $a_3 = \sqrt[3]{a_2} + 1$, and in general $a_{n+1} = \sqrt[3]{a_n} + 1$ for all $n \geq 1$. What is the value of a_5 to the nearest hundredth?

Vocabulary
Concepts
Skills

Graph

Independent

Dependent

Domain

Range

Coefficients

$y = af(bx + c) + d$

Zeros

Intercepts

Asymptotes

Minimum

Maximum

Intersections

Increasing/Decreasing

Global vs. Local Behavior

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Discrete

System of Equations

Solve Equations and Inequalities

Justifying Steps Used

Optional functions for additional or continued study: linear, quadratic, cubic, exponential, rational, parametric, and linear programming